Theoretical physics in transportation

Examination 3, 9. 2. 2015

Abstract

You have 2 hours to answer the questions and prepare the Maple worksheet. I recommend to use comments in your code. As a help you can use your (paper) notebooks, Maple documentation and nothing else. For each part of the examination you have limited time indicated below. In order to proceed to next parts you must accomplish the tasks at least to 50 %.

1 1. Theoretical part (30 minutes)

- Let the force be given by

  $$F_x = -k\,x, \qquad F_y = -k\,y. \tag{1}$$

  Write down the definition of generalized force Q_a in general and then caluclate Q_a for the force (1) in polar coordinates defined by

  $$x = r\,\cos\phi, \qquad y = r\,\sin\phi.$$

- What is hyperbolic critical point? How is hyperbolicity related to the classification of critical points?

- What is the attractor? What is limit cycle?

- Does mathematical pendulum have an attractor? What changes if we introduce the friction?

- Find the Legendre transformation of the function

  $$f(x,y) = x^2 - y^2.$$

2 Easy Maple part (30 minutes)

The Legendre polynomial $P_n(x)$ is function defined by

$$P_n(x) = \frac{1}{2^n\,n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((x^2-1)^n\right). \tag{2}$$

- Plot the first 6 Legendre polynomials ($n = 0, 1, 2 \dots 5$) into a single graph.
- Find the roots of $P_3(x)$ (i.e. solve the equation $P_3(x) = 0$)
- What is the value of P_{23} for $x = 0$?

3 More difficult Maple part (60 minutes)

Consider the Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{\sqrt{x^2 + y^2}}$$

- Write down the Lagrangian in polar coordinates using Maple. If function `csgn` appears in the result (because of the square root), you can eliminate it using

  ```
  subs( [csgn(r(t))=1], ... ),
  ```

  where ... is your Lagrangian.

- Eliminate the time dependence in the Lagrangian, i.e. replace

  $$r(t) \mapsto r, \quad \phi(t) \mapsto \phi, \quad \dot{r}(t) \mapsto v_r, \quad \dot{\phi} \mapsto v_\phi.$$

  To accomplish this, use the `subs` command. Late you may need also the inverse rules, i.e. $r \mapsto r(t)$, etc.

- Derive the Hamiltonian using Maple.
- Solve corresponding Hamilton equations numerically for initial conditions

  $$r(0) = 5, \qquad \phi(0) = 0,$$

  and initial momenta

  $$(p_r(0) = 0, p_\phi(0) = 1), \qquad (p_r(0) = -0.1, p_\phi(0) = 1), \qquad (p_r(0) = 0.1, p_\phi(0) = 1).$$

- Use function `odeplot` to plot the trajectory for each solution, i.e. plot the curve

  $$[r\,\cos\phi, r\,\sin\phi]$$

  for each solution.