Theoretical physics in transportation

Examination 4, 11. 2. 2015

Abstract

You have 2 hours to answer the questions and prepare the Maple worksheet. I recommend to use comments in your code. As a help you can use your (paper) notebooks, Maple documentation and nothing else. For each part of the examination you have limited time indicated below. In order to proceed to next parts you must accomplish the tasks at least to 50 %.

1 1. Theoretical part (30 minutes)

- Is the system described by Hamiltonian

$$\mathcal{H} = \frac{p^2}{2\,m} + \frac{1}{2}\,k\,x\cos t$$

  conservative? Why?

- Simplify the following system of equations using the index notation and Einstein summation convention

$$\begin{aligned}
\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= \frac{\partial f}{\partial x_1}\,x_1^2 + \frac{\partial f}{\partial x_2} x_1\,x_2 + \frac{\partial f}{\partial x_3}\,x_1\,x_3\\
\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= \frac{\partial f}{\partial x_1}\,x_1\,x_2 + \frac{\partial f}{\partial x_2} x_2^2 + \frac{\partial f}{\partial x_3}\,x_2\,x_3\\
\frac{\mathrm{d}^2 x_3}{\mathrm{d}t^2} &= \frac{\partial f}{\partial x_1}\,x_1\,x_3 + \frac{\partial f}{\partial x_2} x_2\,x_3 + \frac{\partial f}{\partial x_3}\,x_3^2
\end{aligned}$$

- What is the difference between *Lagrange's equations* and *Lagrange's equations of the second kind*? Are they physically different? What basic physical law do they represent?

- Show that phase trajectories of the harmonic oscillator are ellipses.

2 Easy Maple part (30 minutes)

The Legendre polynomial $P_n(x)$ is function defined by

$$P_n(x) = \frac{1}{2^n\, n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left((x^2 - 1)^n \right). \tag{1}$$

- Plot the first 6 Legendre polynomials ($n = 0, 1, 2 \dots 5$) into a single graph.
- Find the roots of $P_3(x)$ (i.e. solve the equation $P_3(x) = 0$)
- What is the value of P_{23} for $x = 0$?

3 More difficult Maple part (60 minutes)

Consider the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(x^2 + y^2\right) - \cos(x\, y)$$

- Find the Hamilton's equation
- Plot the solution for initial values

  $$x(0) = 1, \qquad y(0) = 10.$$

- Find the critical points. How many are there?
- Analyze the stability of these points.
- Generate the solutions for the set of initial conditions

  $$x(0) = \cos\frac{2\pi n}{10}, \qquad y(0) = \sin\frac{2\pi n}{10}, \qquad n = 0, 1, \dots 9.$$

  and plot them into a single graph. Annotate the graph with labels $n = 0, n = 1$, etc.