Selected topics in differential geometry

with applications in general relativity

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I Geometry

1. Principal and vector bundles

This chapter is an extract of [KN96], but we are following the notation of [Fec06] more closely.

1.1 Actions on the Lie groups

Definition 1.1.1 A *left action* on a Lie group G is a mapping

$$L : G \times G \mapsto G \qquad , L : (g,h) \mapsto L_g(h) \tag{1.1}$$

which satisfies

$$L_{g_1} \circ L_{g_2} = L_{g_1 g_2}, \qquad L_e = \mathrm{id}, \tag{1.2}$$

where $g, g_1, g_2, h \in G$ are arbitrary elements of the group G, $e \in G$ is the identity of the group and "id" is the identity mapping on G. *Right action* R is defined analogously, the only difference being the requirement

$$R_{g_1} \circ R_{g_2} = R_{g_2} R_{g_1}. \tag{1.3}$$

R On any Lie group, canonical left and right actions are defined by

$$L_g(h) = g\,h, \qquad R_g(h) = h\,g. \tag{1.4}$$

If not stated otherwise, by actions L_g, R_g we will always mean these canonical actions.

Exercise 1.1 Show that mappings (1.4) are indeed left/right actions in the sense of definition

1.1.1. Show that another canonical actions are

$$L_g(h) = h g^{-1}, \qquad R_g(h) = g^{-1} h. \tag{1.5}$$

Definition 1.1.2 Action L is called

- *free*, if $L_{g_1}(h) = L_{g_2}(h)$ implies $g_1 = g_2$;
- *transitive*, if for any $h_1, h_2 \in G$ there exists $g \in G$ such that $h_2 = L_g(h_1)$.

Moreover, a left action induces the *equivalence relation* between elements of G: two elements $g, h \in G$ are *equivalent*, if there exists $m \in G$ such that $L_m(g) = h$.

Similar terminology applies to the right action.

Exercise 1.2 Prove that the condition that action L is free is equivalent to:

$$\text{if } L_g h = h \text{ for some } h, \text{ then } g = e. \tag{1.6}$$

In other words, action L is free if the mapping L_g, for $g \neq e$, does not have any fixed point. ■

1.2 Principal bundles

As we saw in the definition 1.1.2, the existence of an action (say, left) introduces the equivalence relation between the points of the Lie group. We may use standard notation of the set theory, i.e.

$$g \sim h \qquad \equiv \qquad \exists m \in G : h = L_m(g) \tag{1.7}$$

for the relation itself, and symbol

$$[g] = \{h \in G \mid h \sim g\} \tag{1.8}$$

for the equivalence class. In other words, $[g]$ is the set of all points which can be reached from g by the action L. The existence of relation $\sim$ allows us to form a *quotient set* $G/\sim$ which is a set of all (distinct) equivalence classes:

$$G/\sim = \{[g] \mid g \in G\}. \tag{1.9}$$

Mapping $\pi : G \mapsto G/\sim$ defined by

$$\pi : g \mapsto [g] \tag{1.10}$$

is called a *canonical projection*.

For a *transitive group* all elements are equivalent and therefore the factorization collapses entire group into a single point $[e]$. Otherwise, the quotient set consists of “fibers” foliating the entire group. This is the geometrical idea behind the principal G–bundles. Geometrically, we take the manifold M and attach a fiber to each point of M. There is a natural action of the group on each fiber, and all together all fibers (at all points) form a higher dimensional manifold P which is naturally foliated by the fibers.

Definition 1.2.1 *Principal bundle* is a triple (P, G, R, M) where P is a manifold called *total space*, G is a Lie group called *structure group*, R is a free right action of G on P, and M is a manifold such that:

1. M is the quotient space $M = P/\sim$ ($\sim$ is equivalence relation induced by the action R on P) and the canonical projection $\pi : P \mapsto M$ is differentiable;

2. P is *locally trivial*, i.e. for any $x \in M$ there exists a neighborhood U, $x \in U \subset M$, such that $\pi^{-1}(U)$ is isomorphic to $U \times G$,

$$\pi^{-1}(U) \cong U \times G, \tag{1.11}$$

where by isomorphism we mean a mapping $\psi : \pi^{-1}(U) \mapsto U \times G$ which maps $u \in \pi^{-1}(U)$ into $(\pi(u), \varphi(g))$, where $\varphi : \pi^{-1}(U) \mapsto G$ satisfies

$$\varphi(R_g u) = \varphi(u) g. \tag{1.12}$$

R Mapping ψ of definition 1.2.1 is called *local trivialization*. The definition means that locally any element $u \in P$ can be written in the form (x, g), where g is an element of the structure group G. In order to ensure consistency, the right action R_h must map u into an element of the form (u, gh).

II Spinor formalism

$\nabla_A^{A'} \psi_{A'B'C'D'} = 0$

$\phi_{AB} = \phi_{(AB)} + \frac{1}{2} \epsilon_{AB} \phi_X{}^X$

$C_{abcd} = \psi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \text{c.c.}$

2. Summary of the 2-spinor formalism

In this chapter we review basic formulae of the 2-spinor formalism in general relativity. Full treatment with geometrical motivation and detailed discussion can be found in [PR84]. For introduction to spinors at more "practical" level, see [Ste93] and [ODo03]. General theory of spinors (not just 2-spinors) with the physical applications is nicely reviewed in [BT87], geometrical background on connections and Dirac operator can be found in [Fec06].

2.1 Definition of 2-spinors

Spinors are elements of certain vector space equipped with the so-called symplectic form. Their physical relevance stems from the fact that spinors can be related to the tensors on the space-time (in contrast to tensors on abstract manifolds, tensors on the space-time which represent physical quantities are called *world-tensors*)) and that the Lorentz group can act on the space of spinors, i.e. the space of spinors is a representation space of the Lorentz group. In this section we will regard the spinor merely as an element of the aforementioned vector space and its relation to the world-tensors will be discussed subsequently.

Definition 2.1.1 A *spinor space* is a pair (S, ε) two-dimensional vector space S over the field of complex numbers $\mathbb{C}$ equipped with the mapping $\varepsilon : S \times S \mapsto \mathbb{C}$ called *symplectic form* satisfying the following properties:

- linearity in the first argument

$$\varepsilon(\xi + \lambda\,\eta, \cdot) = \varepsilon(\xi, \cdot) + \lambda\,\varepsilon(\eta, \cdot), \qquad \xi, \eta \in S,\ \lambda \in \mathbb{C}; \tag{2.1}$$

- antisymmetry

$$\varepsilon(\xi, \eta) = -\varepsilon(\eta, \xi); \tag{2.2}$$

- non-degeneracy

$$\text{if } \varepsilon(\xi, \eta) = 0 \text{ for all } \eta \in S, \text{ then } \xi = 0. \tag{2.3}$$

Elements of S are called *spinors*.

R Linearity together with antisymmetry implies that ε is in fact linear in both arguments, i.e. *bilinear*.

R Symplectic form introduced in definition 2.1.1 is a bi-linear antisymmetric mapping from $S \times S$ to $\mathbb{C}$, so it is a *2-form* on S, i.e. $\varepsilon \in \Lambda^2(S)$. Since the dimension of space of p-forms on vector space L of dimension n is, in general,

$$\dim \Lambda^p(L) = \binom{n}{p}, \tag{2.4}$$

the space of two-forms on two-dimensional spinor space is $\dim \Lambda^2(S) = 1$. That is, for given (S, ε), *all* two-forms on S have the form $\lambda\, \varepsilon$, $\lambda \in \mathbb{C}$. This fact will play a crucial role in algebraic decomposition of the spinors.

R From an algebraical point of view, the most important property of the symplectic form ε is its non-degeneracy. This property ensures that for any non-zero spinor $\xi \in S$ there exists a spinor $\eta \in S$ such that $\varepsilon(\xi, \eta) \neq 0$. This is somewhat analogous to the *metric tensor* g on a vector space L, which is also non-degenerate and hence admits an inverse g^{-1} such that $g^{ab} g_{bc} = \delta^a_c$. Metric g introduces an isomorphism between spaces L and $L^\star$, i.e. between the vector space and its dual. The same holds for ε: it introduces an isomorphism between S and $S^\star$ and hence we are allowed to raise and lower the indices using ε and its inverse ε^{-1}. The crucial difference between metric g and symplectic form ε is that the former is symmetric, while the latter is antisymmetric. This, in turn, means that the "scalar product" provided by ε is *anticommutative*.

Definition 2.1.2 Given a spinor space (S, ε) and any two spinors $\xi, \eta \in S$, number $\varepsilon(\xi, \eta)$ is called the *symplectic product* of spinors ξ and η. Sometimes we use the notation

$$[\xi, \eta] \equiv \varepsilon(\xi, \eta). \tag{2.5}$$

R By definition 2.1.1 of the symplectic form, the symplectic product is antisymmetric, i.e. $[\xi, \eta] = -[\eta, \xi]$.

Exercise 2.1 Prove the following statement: two spinors $\xi, \eta \in S$ are linearly dependent if, and only if, $[\xi, \eta] = 0$.

(*Hint*: S is 2-dimensional. Direction $\rightarrow$: linear dependence means $\eta = \lambda\, \xi$, antisymmetry then implies $[\xi, \eta] = 0$. Direction $\leftarrow$ (by contradiction): assume $[\xi, \eta] = 0$ with ξ, η independent (i.e., constituting a basis of S), non-degeneracy implies the existence of ζ such that $[\xi, \zeta] \neq 0$, but ζ must be combination of ξ and η, so $[\xi, \zeta] \neq 0$ is impossible.) ■

Definition 2.1.3 A linear mapping $\phi : S \mapsto S$ on the spinor space (S, ε) is called *symplectomorphism* if it preserves the symplectic form in the sense

$$[\phi(\xi), \phi(\eta)] = [\xi, \eta] \qquad \text{for any } \xi, \eta \in S. \tag{2.6}$$

The group of all symplectomorphisms on S is called the *symplectic group* $\mathrm{Sp}(S)$.

It will turn out that for vector space S of dimension 2, the symplectic group $\mathrm{Sp}(S)$ is isomorphic to group $\mathbb{SL}(2, \mathbb{C})$ which is a covering group of the *Lorentz group*.

2.2 Abstract index notation

In what follows we will occasionally employ the geometrical, "index-free" notation but most of the time we employ the *abstract index notation* developed by Penrose (see [PR84] for a detailed discussion). In this notation we introduce an infinite set of copies of original vector space S and label them by the elements of the set of indices. For spinors, the set of indices is

$$\mathscr{I} = \{A, B, C, \dots X, Y, \dots\}, \tag{2.7}$$

where we usually use only a small number of these symbols, but in principle the set of indices must be infinite in order to allow for arbitrarily complicated expressions. Now, for any $\alpha \in \mathscr{I}$ we define the vector space

$$S^\alpha = \{(\xi, \alpha) | \xi \in S\}, \qquad \alpha \in \mathscr{I}. \tag{2.8}$$

Thus, S^α is a (n isomorphic) copy of original vector space S but each vector is labeled by index α. Instead of writing (ξ, α) we will use the symbol ξ^α.

We emphasize that the index $\alpha \in \mathscr{I}$ here does not acquire numerical values like in, e.g., relativity, where in the expression v^μ we automatically mean that μ acquires values $\mu = 0, 1, 2, 3$. Instead, α is just a marker indicating which vector space spinor ξ^α belongs to. Thus, $\xi^\alpha \in S^\alpha$ but $\xi^\alpha \neq S^\beta$, provided that $\alpha \neq \beta$; for example, $\xi^A \in S^A$, but $\xi^A \neq S^B$. In particular, we cannot add spinors ξ^A and η^B, because they are elements of different vector spaces. Moreover, we have canonical isomorphisms between spaces S^α and S^β so that vector ξ^α has its unique counter-part in S^β.

Associated with each S^α is its *dual space* which is the space of linear functionals on S^α. Dual space of S is denoted by $S^\star$ and its elements, *dual spinors*, are mappings $\zeta : S \mapsto \mathbb{C}$. The isomorphic copies of $S^\star$ will be denoted by S_α, $\alpha \in \mathscr{I}$, i.e.

$$S_A, S_B, S_C, \dots, S_X, S_Y, \dots \tag{2.9}$$

Elements of S_α will carry the same index, i.e. $\zeta_A \in S_A, \zeta_B \in S_B$ etc. An element $\zeta_\alpha \in S_\alpha$ is then a linear mapping

$$\zeta_\alpha : S^\alpha \mapsto \mathbb{C} : \xi^\alpha \mapsto \zeta_\alpha(\xi^\alpha) \in \mathbb{C}. \tag{2.10}$$

It is clear that, say, ζ_A cannot act on any spinor ξ^B because ζ_A is a linear functional on S^A, while ξ^B belongs to S^B. Thanks to the linearity of dual spinors it is then redundant to write the brackets. Hence, we can omit them and write $\zeta_A \xi^A$ instead of $\zeta_A(\xi^A)$. No confusion can arise because the indices keep track of which functional is acting on what.

This is the main upshot of the abstract index notation (which equally applies to tensors as well). On the one hand, it is coordinate-free like usual geometrical language. We are not talking about the components of spinors (tensors) with respect some basis, we are talking about the objects themselves. Hence, in the abstract index notation $\zeta_A(\xi^A)$ is equivalent to "geometer's notation" $\zeta(\xi)$. At the same time, the abstract index notation resembles the "relativist's" index notation, where ζ_A stands for the pair of components ζ_0 and ζ_1. Hence, abstract index notation has the full advantage of index notation which is preferred for practical calculations, but it is also manifestly basis-independent like the index-free notation.

These considerations readily extend to all tensorial operations. The tensor product of multiple copies of S and $S^\star$ in the index-free notation is

$$T^P_q(S) = \underbrace{S^\star \otimes \cdots \otimes S^\star}_{p} \otimes \underbrace{S \otimes \cdots \otimes S}_{q}. \tag{2.11}$$

The set $T^p_q(S)$ is in fact the vector space of tensors of rank (p,q) over the vector space S. In the abstract index notation we denote

$$S^{\alpha\dots\beta}{}_{\gamma\dots\delta} = S^\alpha \otimes \cdots \otimes S^\beta \otimes S_\gamma \otimes \cdots \otimes S_\delta, \tag{2.12}$$

where all the indices $\alpha,\dots,\beta,\gamma,\dots\delta \in \mathscr{I}$ must be *different*. Elements of $S^{\alpha\dots\beta}{}_{\gamma\dots\delta}$ will be denoted by $\xi^{\alpha\dots\beta}{}_{\gamma\dots\delta}$. So, for example, spinor $\xi^{AB}{}_C$ of *valence* $(2,1)$ is a mapping

$$\xi^{AB}{}_C : S_{AB}{}^C \mapsto \mathbb{C} : (\alpha_A, \beta_B, \eta^C) \mapsto \xi^{AB}{}_C\, \alpha_A\, \beta_B\, \eta^C. \tag{2.13}$$

We can also define the *tensor product* of spinors. For the two spinors of arbitrary valence

$$\xi^{\alpha\dots\beta}{}_{\gamma\dots\delta}, \qquad \eta^{\rho\dots\sigma}{}_{\mu\dots\lambda} \tag{2.14}$$

we define their tensor product as a spinor

$$(\xi\otimes\eta)^{\alpha\dots\beta\rho\dots\lambda}{}_{\gamma\dots\delta\mu\dots\lambda} \equiv \xi^{\alpha\dots\beta}{}_{\gamma\dots\delta}\eta^{\rho\dots\sigma}{}_{\mu\dots\lambda} \tag{2.15}$$

defined by its action on appropriate number of spinors and dual spinors as

$$\begin{aligned}(\xi\otimes\eta)^{\alpha\dots\beta\rho\dots\lambda}{}_{\gamma\dots\delta\mu\dots\lambda}(u_\alpha,\dots,v_\beta,\tilde u_\rho,\dots,\tilde v_\lambda, w^\beta,\dots,z^\gamma,\tilde w^\mu,\dots,\tilde w^\lambda) &= \\ = \left(\xi^{\alpha\dots\beta}{}_{\gamma\dots\delta}\, u_\alpha \dots v_\beta\, w^\beta \dots z^\gamma\right)\left(\eta^{\rho\dots\sigma}{}_{\mu\dots\lambda}\, \tilde u_\alpha \dots \tilde v_\beta\, \tilde w^\beta \dots \tilde z^\gamma\right),\end{aligned} \tag{2.16}$$

where

$$u,\dots,v,\tilde u,\dots,\tilde v \in S, \qquad w,\dots,z,\tilde w,\dots,\tilde s \in S^\star. \tag{2.17}$$

One of the advantages of the index formalism is a possibility to express several symmetry operations in a compact way. As it is usual, we define the *symmetric part* and *antisymmetric* part of a spinor $\xi_{\alpha_1\dots\alpha_p}$, respectively, by

$$\xi_{(\alpha_1\dots\alpha_p)} = \frac{1}{p!}\sum_\sigma \xi_{\sigma_1\dots\sigma_p}, \qquad \xi_{[\alpha_1\dots\alpha_p]} = \frac{1}{p!}\sum_\sigma \operatorname{sign}\sigma\, \xi_{\sigma_1\dots\sigma_p}, \tag{2.18}$$

where the sum goes through all permutations σ of the set $\{\alpha_1,\dots,\alpha_p\}$ and $\operatorname{sign}\sigma$ is the sign of σ. In particular, we have

$$\xi_{(\alpha\beta)} = \frac{1}{2}\left(\xi_{\alpha\beta} + \xi_{\beta\alpha}\right), \qquad \xi_{[\alpha\beta]} = \frac{1}{2}\left(\xi_{\alpha\beta} - \xi_{\beta\alpha}\right). \tag{2.19}$$

The space of symmetric (antisymmetric) spinors will be denoted by $S_{(\alpha\beta)}$ ($S_{[\alpha\beta]}$).

R So far, we said that the abstract indices $A,B,\dots$ form a set of indices $\mathscr{I}$ and arbitrary elements of this set were denoted by $\alpha,\beta,\dots$. The purpose was to emphasize that in the formulae above hold for any abstract indices from the set $\mathscr{A}$. However, in what follows we simplify the notation and use particular abstract indices. That is, instead of writing S^α, where $\alpha\in\mathscr{A}$, we will write simply S^A and it will be automatically understood that arbitrary abstract index can be substituted.

An important example of a spinor is the *identity mapping*. In non-abstract formalism we use the so-called *Kronecker symbol* δ^B_A which is a unit matrix. In the abstract index formalism, we regard the Kronecker symbol as a *mapping*

$$\delta^A_B : S_A \otimes S^B \mapsto \mathbb{C} \tag{2.20}$$

defined by

$$\delta^A_B : (\xi_A, \eta^B) \mapsto \delta^A_B\, \xi_A\, \eta^B = \xi_A\, \eta^A. \tag{2.21}$$

This of course resembles usual relation for Kronecker delta, but the interpretation is slightly different, since δ^A_B is now not a matrix (set of numbers) bu an abstract mapping which maps a spinor and a dual spinor to a complex number (equal to the contraction of the two spinors).

Exercise 2.2 Show formally that δ^A_B can be regarded as the aforementioned canonical isomorphism $S^B \mapsto S^A$ or $S_A \mapsto S_B$ and the following relations hold:

$$\delta^A_B : \xi^B \mapsto \xi^A, \qquad \delta^A_B : \eta_A \mapsto \eta_B, \tag{2.22}$$

justifying the name "identity map". ■

An example of the antisymmetric spinor is the symplectic form. In the abstract index notation, ε_{AB} is regarded as an element of the space $S_{[AB]}$ and its action on a pair of (univalent) spinors is

$$\varepsilon \in S_{[AB]} : S^{AB} \mapsto \mathbb{C} : (\xi^A, \eta^B) \mapsto \varepsilon_{AB}\xi^A\eta^B \in \mathbb{C}. \tag{2.23}$$

Its antisymmetry is expressed by the relation $\varepsilon_{AB} = -\varepsilon_{BA}$.

Exercise 2.3 Show that ε_{AB} can be regarded as a canonical isomorphism of spaces S^A and S_B defined by

$$\varepsilon_{AB} : S^A \mapsto S_B : \xi^A \mapsto \varepsilon_{AB}\,\xi^A \in S_B. \tag{2.24}$$

■

Thanks to the isomorphism introduced in the exercise 2.3 we can lower the index of a spinor ξ^A according to the rule

$$\xi_B = \varepsilon_{AB}\,\xi^A. \tag{2.25a}$$

Since this mapping is an isomorphism, it has an inverse which is defined by

$$\xi^A = \varepsilon^{AB}\xi_B, \tag{2.25b}$$

where $\varepsilon^{AB} \in S^{[AB]}$ is a mapping satisfying

$$\varepsilon^{AB}\varepsilon_{BC} = \delta^A_C. \tag{2.25c}$$

This is very similar to raising/lowering indices via metric tensor in tensor algebra, but here, due to antisymmetry of the symplectic form, one must be careful about the order of the indices. In particular, $\varepsilon_{AB}\xi^B = -\varepsilon_{BA}\xi^B = -\xi_A$.

2.3 Spin basis

Although for general purposes it is useful to use abstract index notation and, hence, avoid the possible dependence of the formulae on a particular choice of the basis. Nevertheless, sometimes we *need* to introduce a particular basis. This happens, for example, if the problem we study has some symmetry so that the basis respecting this symmetry can simplify relevant equations. Naive counting of degrees of freedom shows that the spinor space S has the same dimensionality as the tangent space of the space-time. The former has complex dimension 2 while the latter has real dimension 4. Thus, as real vector spaces, they both have dimension 4 and hence are isomorphic. Later we will show that also the Lorentz group acts on the spinor space naturally which makes the correspondence between world-tensors and spinors interesting.

In the space-time tangent space we often choose an orthonormal basis. By exercise 2.1 we know that the basis of S cannot be orthonormal, i.e. we cannot require $[\xi, \eta] = 0$ because that would immediately mean that ξ and η are linearly dependent and, hence, would not form a basis. In other words, for any two basis spinors their symplectic product must be non-zero. Thus, the next most natural thing is to normalize them to unity.

Definition 2.3.1 Two spinors $o, \iota \in S$ form a *spin basis* if

$$[o, \iota] = 1. \tag{2.26}$$

R In the abstract index notation, condition (2.26) reads

$$\varepsilon_{AB}\, o^A\, \iota^B = 1, \tag{2.27}$$

or, using the rule (2.25a),

$$o_A\, \iota^A = -o^A\, \iota_A = 1 \tag{2.28}$$

R Instead of spin basis, the term *spin dyad* is often employed. This is consistent with calling a basis of four-dimensional vector space the *tetrad*.

$$\nabla_A^{A'}\psi_{A'B'C'D'} = 0$$

$$\phi_{AB} = \phi_{(AB)} + \frac{1}{2}\epsilon_{AB}\phi_X{}^X$$

$$C_{abcd} = \psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \text{c.c.}$$

3. Newman–Penrose formalism

3.1 Newman–Penrose null tetrad

Definition 3.1.1 A *Newman–Penrose null tetrad* is a four-tuple of vectors $(\ell^a, n^a, m^a, \bar{m}^a)$ which satisfy

$$\ell^a n_a = 1, \qquad m^a \bar{m}_a = -1, \tag{3.1}$$

and all remaining contractions are zero (in particular, all vectors are null); vector $\bar{m}^a$ is a complex conjugate of m^a, while ℓ^a and n^a are real.

R In the Newman–Penrose tetrad, the metric tensor can be decomposed as

$$g_{ab} = \ell_a n_b + \ell_b n_a - m_a \bar{m}_b - m_b \bar{m}_a. \tag{3.2}$$

Definition 3.1.2 Given a Newman–Penrose null tetrad, covariant derivatives associated with vectors forming the null tetrad are denoted by

$$D = \ell^a \nabla_a, \qquad \Delta = n^a \nabla_a, \qquad \delta = m^a \nabla_a, \qquad \bar{\delta} = \bar{m}^a \nabla_a. \tag{3.3}$$

R The covariant derivative can be decomposed, with the help of (3.2), as

$$\nabla_a = g_a^b \nabla_b = \ell_a \Delta + n_a D - m_a \bar{\delta} - \bar{m}_a \delta. \tag{3.4}$$

Exercise 3.1 Show that if (o^A, ι^A) is a spin basis, then vectors defined by

$$\ell^a = o^A \bar{o}^{A'}, \qquad n^a = \iota^A \bar{\iota}^{A'}, \qquad m^a = o^A \bar{\iota}^{A'}, \qquad \bar{m}^a = \iota^A \bar{o}^{A'}, \tag{3.5}$$

comprise a null tetrad. ■

3.2 Spin coefficients and connection

The *spin coefficients* are defined as follows (table adopted from [Ste93]):

∇	$o^A \nabla o_A$	$o^A \nabla \iota_A$	$\iota^A \nabla \iota_A$
D	κ	ε	π
Δ	τ	γ	ν
δ	σ	β	μ
$\bar{\delta}$	ρ	α	λ

Thus, for example, $\kappa = o^A D o_A$. The definitions of the spin coefficients imply the following relations for covariant derivatives of basis spinors:

$$D o_A = \varepsilon\, o_A - \kappa\, \iota_A, \tag{3.6a}$$
$$D \iota_A = \pi\, o_A - \varepsilon\, \iota_A, \tag{3.6b}$$
$$\delta o_A = \beta\, o_A - \sigma\, \iota_A, \tag{3.6c}$$
$$\delta \iota_A = \mu\, o_A - \beta\, \iota_A, \tag{3.6d}$$
$$\Delta o_A = \gamma o_A - \tau\, \iota_A, \tag{3.6e}$$
$$\Delta \iota_A = \nu\, o_A - \gamma \iota_A, \tag{3.6f}$$
$$\bar{\delta} o_A = \alpha\, o_A - \rho\, \iota_A, \tag{3.6g}$$
$$\bar{\delta} \iota_A = \lambda\, o_A - \alpha\, \iota_A. \tag{3.6h}$$

Tensor equivalents of the definitions of the spin coefficients are

$$\kappa = m^a D \ell_a = o^A D o_A, \qquad \tau = m^a \Delta \ell_a = o^A \Delta o_A, \tag{3.7a}$$
$$\sigma = m^a \delta \ell_a = o^A \delta o_A, \qquad \rho = m^a \bar{\delta} \ell_a = o^A \bar{\delta} o_A, \tag{3.7b}$$
$$\pi = n^a D \bar{m}_a = \iota^A D \iota_A, \qquad \nu = n^a \Delta \bar{m}_a = \iota^A \Delta \iota_A, \tag{3.7c}$$
$$\lambda = n^a \bar{\delta} \bar{m}_a = \iota^A \bar{\delta} \iota_A, \qquad \mu = n^a \delta \bar{m}_a = \iota^A \delta \iota_A, \tag{3.7d}$$

$$\varepsilon = \frac{1}{2}\left[n^a D \ell_a - \bar{m}^a D m_a\right] = \iota^A D o_A, \tag{3.7e}$$
$$\beta = \frac{1}{2}\left[n^a \delta \ell_a - \bar{m}^a \delta m_a\right] = \iota^A \delta o_A, \tag{3.7f}$$
$$\gamma = \frac{1}{2}\left[n^a \Delta \ell_a - \bar{m}^a \Delta m_a\right] = \iota^A \Delta o_A, \tag{3.7g}$$
$$\alpha = \frac{1}{2}\left[n^a \bar{\delta} \ell_a - \bar{m}^a \bar{\delta} m_a\right] = \iota^A \bar{\delta} o_A, \tag{3.7h}$$

The spin coefficients (3.7) allows one to express the directional covariant derivatives of the tetrad vectors; these are called the *transport equations*:

$$D\ell^a = (\varepsilon + \bar{\varepsilon})\,\ell^a - \bar{\kappa}\, m^a - \kappa\, \bar{m}^a, \tag{3.8a}$$
$$\Delta\ell^a = (\gamma + \bar{\gamma})\,\ell^a - \bar{\tau}\, m^a - \tau\, \bar{m}^a, \tag{3.8b}$$
$$\delta\ell^a = (\bar{\alpha} + \beta)\,\ell^a - \bar{\rho}\, m^a - \sigma\, \bar{m}^a, \tag{3.8c}$$
$$Dn^a = -\,(\varepsilon + \bar{\varepsilon})\, n^a + \pi\, m^a + \bar{\pi}\, \bar{m}^a, \tag{3.8d}$$
$$\Delta n^a = -\,(\gamma + \bar{\gamma})\, n^a + \nu\, m^a + \bar{\nu}\, \bar{m}^a, \tag{3.8e}$$
$$\delta n^a = -(\bar{\alpha} + \beta) n^a + \mu\, m^a + \bar{\lambda}\, \bar{m}^a, \tag{3.8f}$$
$$Dm^a = \bar{\pi}\,\ell^a - \kappa\, n^a + (\varepsilon - \bar{\varepsilon})\, m^a, \tag{3.8g}$$
$$\Delta m^a = \bar{\nu}\,\ell^a - \tau\, n^a + (\gamma - \bar{\gamma})\, m^a, \tag{3.8h}$$
$$\delta m^a = \bar{\lambda}\,\ell^a - \sigma\, n^a + (\beta - \bar{\alpha})\, m^a, \tag{3.8i}$$
$$\bar{\delta} m^a = \bar{\mu}\,\ell^a - \rho\, n^a + \left(\alpha - \bar{\beta}\right) m^a. \tag{3.8j}$$

$\nabla_A^{A'} \psi_{A'B'C'D'} = 0$

$\phi_{AB} = \phi_{(AB)} + \frac{1}{2} \epsilon_{AB} \phi_X{}^X$

$C_{abcd} = \psi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \text{c.c.}$

4. Optical scalars

As a first application of the spinor and Newman–Penrose formalism we discuss the so-called *optical scalars*. It is a set of scalar quantities which characterize the behavior of null geodesics in curved space-times. As emphasized before, in the analysis of the geometry of a certain space-time, it is important to distinguish the coordinate-dependent behavior, like the singularity of the Schwarzschild metric on the horizon, and behavior which is independent of coordinates. The Newman–Penrose null tetrad is not unique and at each point of the space-time we can choose it in an arbitrary way. In this sense, the choice of the Newman–Penrose tetrad is as arbitrary as the choice of the coordinates. However, in many cases, there is a preferred choice of a tetrad. For example, the principal null directions of the Weyl tensor can be used to define a null tetrad. In particular, in type D space-times, which has two principal null directions, it is natural to define the spin basis in such a way that the basis spinors o^A and ι^A define the two principal null directions,

$$\ell^a = o^A \bar{o}^{A'}, \qquad n^a = \iota^A \bar{\iota}^{A'}. \tag{4.1}$$

With this choice, the only non-vanishing Weyl scalar is Ψ_2 which simplifies the analysis significantly.

Nevertheless, even in the case of algebraically general space-times, we often study some given null congruence. It may be, for example, identified with the direction of propagation of gravitational radiation. In such a case, we naturally choose one of the vectors of the null tetrad (ℓ^a or n^a) to be tangent to that congruence. Then, there is still some freedom in the choice of remaining Newman–Penrose vectors, but already the fact that ℓ^a (or n^a) is a null geodesic simplifies some of the spin coefficients.

The study of null geodesics is of particular interest in general relativity. We know that null rays separate the regions of events which can be causally related (i.e. they are time-like separated) and the region of events where no causal relation is possible (i.e. they are space-like separated). In this sense, null geodesics define the causal structure of the space-time. In order to understand the causal structure, it is instructive to analyze the behavior of null geodesics. For example, on the event horizon of a black hole, all null geodesics are converging (hence, the event horizon is a trapped surface) which is an indication of the singularity below the event horizon.

There are basically three kinds of behavior of null geodesics: convergence/divergence, shear and twist. The purpose of this chapter is to explain what these terms mean and how they are conveniently encoded in the Newman–Penrose spin coefficients. In order to see this, we will employ the spinor formalism.

4.1 Expansion, shear, twist

Consider an affinely parametrized null geodesic congruence with the tangent vector ℓ^a; thus, $D\ell^a =$ in the Newman–Penrose formalism. Now, choose a (three-dimensional) hypersurface Σ which is everywhere transversal to the congruence (i.e. not parallel). Then, at each point $P \in \Sigma$ we can choose a spinor o^A such that $\ell^a = o^A \bar{o}^{A'}$ and propagate this spinor along ℓ^a by the condition $Do_A = 0$. By (3.6), the latter condition implies

$$\varepsilon = 0, \qquad\qquad \kappa = 0. \tag{4.2}$$

In addition, at each point $P \in \Sigma$ we complete o^A to the spin basis (o^A, ι^A) an propagate the spinor ι^A along the congruence by the condition $D\iota^A = 0$ which, again by 3.6, implies

$$\pi = 0. \tag{4.3}$$

R Notice that although we assume $D\ell^a = 0$, this does not mean that $\varepsilon = 0$, for this we indeed need the condition $Do_A = 0$. Which weaker condition is implied by $D\ell^a = 0$?

Now, at each point of the space-like hypersurface Σ we choose a vector z^a which is orthogonal to ℓ^a. We propagate it along ℓ^a by the condition $\pounds_\ell z^a = 0$, i.e. we require that z^a be Lie-constant along ℓ^a. In other words, we choose a vector field z^a orthogonal to ℓ^a on Σ and Lie drag it along the congruence, so that the field z^a commutes with ℓ^a at each point,

$$\pounds_\ell z^a = [\ell, z]^a = 0. \tag{4.4}$$

For a given congruence ℓ^a, vector z^a which commutes with ℓ^a is called the *connecting vector*, and it can be interpreted like a vector which points to a neighboring geodesic of the congruence. Commutation of vectors ℓ^a and z^a is expressed as $Dz^a - z^b\nabla_b z^a = 0$. Then, z^a remains orthogonal to ℓ^a at each point of the congruence, since we have

$$\pounds_\ell(\ell_a z^a) = D(\ell_a z^a) = z^a \underbrace{D\ell_a}_{0} + \underbrace{\ell_a D z^a}_{\ell_a z^b \nabla_b \ell^a} = \ell_a z^b \nabla_b \ell^a = \frac{1}{2} z^b \nabla_b \underbrace{(\ell^a \ell_a)}_{0} = 0. \tag{4.5}$$

Hence, we have a null geodesic congruence ℓ^a and connecting vector z^a which is
- everywhere orthogonal to ℓ^a;
- Lie constant along z^a, i.e. $[\ell, z]^a = 0$.

Having defined a spin basis (o^A, ι^A), we can define a Newman–Penrose null tetrad via relations introduced in the exercise 3.1, page 17. Since the Newman–Penrose tetrad is a basis of each tangent space of the space-time, we can expand any vector field z^a as

$$z^a = A\,\ell^a + B\,n^a + \bar{z} m^a + z \bar{m}^a. \tag{4.6}$$

Here, A, B and z are undetermined coefficients, A, B being real and $z, \bar{z}$ being mutually complex conjugated (this relation between z and $\bar{z}$ is necessary in order to make z^a real). However, condition $z^a \ell_a = 0$ immediately implies $B = 0$.

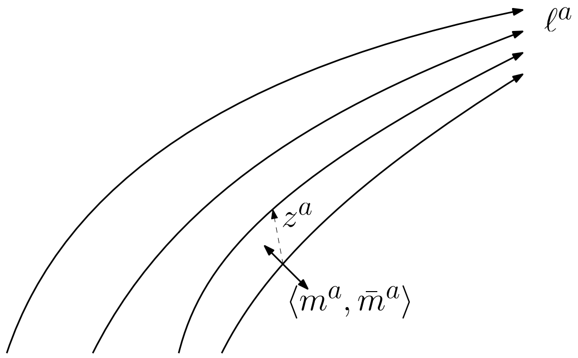


Figure 4.1: Connecting vector z^a can be projected onto the screen which is a vector space spanned by m^a and $\bar{m}^a$.

Let us discuss the geometrical interpretation of (4.6) briefly. Vector field ℓ^a is a congruence we study and we would like to understand its behavior in terms of geometrically understandable quantities. In order to get an intuitive picture, we introduced vector z^a which is everywhere orthogonal to ℓ^a. We called it "connecting vector" because its geometrical interpretation is that given a space-time point x^μ, the neighboring geodesics is $\ell^\mu(x') = \ell^\mu(x) + z^\mu$. Hence, z^a describes the deviation of two neighboring geodesics of the congruence. Now, in order to understand the geometrical properties of ℓ^a, we study how this connecting vector z^a behaves. We expand it into a null tetrad but because of its orthogonality to ℓ^a it has only ℓ^a and m^a, $\bar{m}^a$ components. By definition 3.1.1, m^a and $\bar{m}^a$ are orthogonal to ℓ^a. Moreover, they form a basis of space-like subspace of the tangent space at any space-time point. Hence, we can imagine z^a as being projected onto the screen which is perpendicular to ℓ^a, see Fig. 4.1.

Coordinates in the plane m^a and $\bar{m}^a$ have been denoted by $\bar{z}$ and z. Since vectors m^a and $\bar{m}^a$ comprise the basis of *complexified* vector space, we can regard them as complex coordinates in the complex plane, identifying

$$z = x + \mathrm{i}\,y, \qquad \bar{z} = x - \mathrm{i}\,y. \tag{4.7}$$

Hence, we are interested in the time evolution of coordinates z and $\bar{z}$ along the null geodesics ℓ^a. In order to find this evolution, we calculate Dz^a, using the commutation of ℓ^a and z^a:

$$Dz^a = z^b \nabla_b \ell^a. \tag{4.8}$$

Now, using (4.6) (with $B = 0$, as explained), we find

$$Dz^a = A\, D\ell^a + \bar{z}\delta\ell^a + z\bar{\delta}\ell^a. \tag{4.9}$$

III Asymptotic symmetries

$$\nabla^{A'}_{A}\psi_{A'B'C'D'} = 0$$

$$\phi_{AB} = \phi_{(AB)} + \frac{1}{2}\epsilon_{AB}\,\phi_X{}^X$$

$$C_{abcd} = \psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \text{c.c.}$$

5. Asymptotic flatness

In order to understand the structure of a physical theory it is often important to study the so-called *isolated systems*. This allows one to study particular features of the theory using the simple models without necessity to have a full solution describing entire universe in detail [Ger77]. In general relativity it is difficult to say whether a given system is isolated or not, since the field of interest, the metric g_{ab}, is not propagating on a given background, but also defines the background itself. Nevertheless, the basic intuition is that if gravitating sources are localized in a finite domain, one can "go" to infinity where the gravitational field, i.e. the curvature of the spacetime, is supposed to decay and the metric of the spacetime should approach the flat Minkowski metric far from the isolated source. In other words, we expect that spacetimes representing isolated sources are *asymptotically flat*, i.e. flat far from the sources.

It is not obvious, though, how to decide whether a given spacetime is asymptotically flat in the above sense. What does "infinity" mean? How do we localize it in a given coordinate system? For example, for Schwarzschild black hole in usual coordinates it is clear that for $r \to \infty$ the Schwarzschild metric approaches the Minkowski metric. But the choice of the coordinates is arbitrary. Have we defined another coordinate $l = 1/r$, then the infinity would be located at $l \to 0$. If we compactify the coordinate r by $R = \arctan r$, the infinity is located at $R \to \pi/2$, etc.

There are essentially two ways how to deal with this problem. The first one is to restrict the coordinate system by additional conditions. In the original approach of Bondi et al. [BBM62; Sac62] the coordinates $x^\mu = (u, r, x^2, x^3)$ are adapted to a family of null hypersurface labeled by coordinate u and r is a parameter along the generators of these hypersurfaces. If the metric in these coordinates acquires a form which approaches the Minkowski metric for $r \to \infty$, spacetime is said to be asymptotically flat.

Here we discuss in detail another approach developed by Penrose [Pen65; Pen11] that is based on the conformal techniques. We essentially follow the textbook [Ste93]. Deeper insight into conformal structure of spacetimes can be found in [Ger77; PR86]. A useful review paper of conformal infinity and issues related to numerical relativity is [Fra00].

5.1 Motivation: Minkowski spacetime

Let us consider Minkowski metric in spherical coordinates,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\Sigma^2, \qquad \mathrm{d}\Sigma^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2. \tag{5.1}$$

The ranges of the coordinates are

$$t \in (-\infty,\infty), \qquad r \in [0,\infty), \qquad \theta \in [0,\pi], \qquad \phi \in [0,2\pi). \tag{5.2}$$

Because of factor r^2, metric (5.1) has a coordinate singularity for $r \to \infty$. First we transform the metric to double-null coordinates u, v defined by

$$u = t - r, \qquad v = t + r, \qquad t = \frac{1}{2}(v+u), \qquad r = \frac{1}{2}(v-u), \tag{5.3}$$

so that

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v - \frac{1}{4}(v-u)^2\mathrm{d}\Sigma^2. \tag{5.4}$$

Both $u, v \in (-\infty,\infty)$, but because of $r \geq 0$ there is a constraint

$$v \geq u. \tag{5.5}$$

We compactify the ranges of u and v by introducing new coordinates

$$U = \arctan u, \qquad V = \arctan v, \tag{5.6}$$

in which the metric acquires the form

$$\mathrm{d}s^2 = \frac{1}{\cos^2 U \cos^2 V}\left(\mathrm{d}U\,\mathrm{d}V - \frac{1}{4}\sin^2(V-U)\,\mathrm{d}\Sigma^2\right). \tag{5.7}$$

The ranges are now

$$-\frac{\pi}{2} \leq U \leq \frac{\pi}{2}, \qquad -\frac{\pi}{2} \leq V \leq \frac{\pi}{2}, \qquad V \geq U. \tag{5.8}$$

Metric (5.7) still has a singularity at infinity, i.e. for values $U, V = \pm\pi/2$. An important observation is that in this form, the singularity is a common factor in front of the expression which is perfectly regular at infinity. Penrose's trick is therefore to introduce a new, *unphysical* metric by a *conformal rescaling* of the physical metric. Conformal rescaling is a transformation

$$\widehat{\mathrm{d}s}^2 = \Omega^2\,\mathrm{d}s^2 \tag{5.9}$$

with the *conformal factor*

$$\Omega^2 = 4\cos^2 U \cos^2 V, \tag{5.10}$$

so that the unphysical metric becomes

$$\widehat{\mathrm{d}s}^2 = 4\,\mathrm{d}U\,\mathrm{d}V - \sin^2(V-U)\,\mathrm{d}\Sigma^2. \tag{5.11}$$

This metric does not represent the Minkowski spacetime anymore. However, conformal transformations preserve the angles and norms of null vectors. Therefore, it preserves the null cones and, hence this new spacetime has the same causal structure as the Minkowski spacetime. The advantage

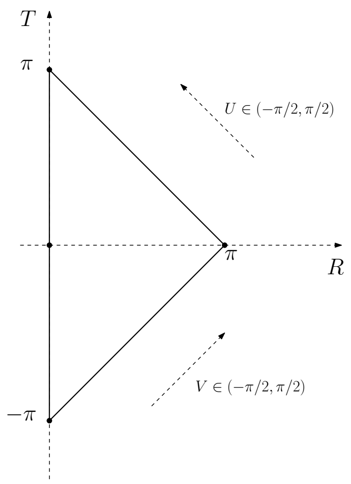


Figure 5.1: Ranges of coordinates in the unphysical Minkowski spacetime

is that metric (5.11) is regular everywhere in the spacetime and, in particular at infinity and, thus, infinity of Minkowski spacetime is a set of well-defined points in the new unpysical spacetime.

It is convenient to switch from the double-null coordinates U,V back to temporal and radial coordinates T,R by

$$T = V + U, \qquad R = V - U, \tag{5.12}$$

in which the metric reads

$$\widehat{\mathrm{d}s}^2 = \mathrm{d}T^2 - \mathrm{d}R^2 - \sin^2 R\,\mathrm{d}\Sigma^2, \tag{5.13}$$

and the ranges of coordinates are

$$-\pi \leq T \leq \pi, \qquad 0 \leq R \leq \pi. \tag{5.14}$$

Hence, in addition to the Minkowski spacetime (M,g) equipped with the line element $\mathrm{d}s^2$ we introduced the unphysical spacetime $(\hat{M},\hat{g})$ with the line element $\widehat{\mathrm{d}s}^2$ and a mapping

$$\psi : M \mapsto \hat{M} \tag{5.15}$$

whose coordinate expression is

$$\psi(t,r,\theta,\phi) = (\arctan(t+r) + \arctan(t-r),\ \arctan(t+r) - \arctan(t-r),\ \theta,\phi)\,. \tag{5.16}$$

The two metrics are related by a conformal rescaling, i.e.

$$\psi^*\hat{g} = \Omega^2 g, \tag{5.17}$$

where ψ^* is a pull-back from $\hat{M}$ to M. Since the range of coordinates (U,V) is finite, we can represent the unphysical spacetime in a finite region as in Figure 5.1.

Exercise 5.1 Show that (timelike) lines of constant r in Minkowski spacetime mapped to $\hat{M}$ are curves which for $t \to -\infty$ start at the point $(T,R) = (-\pi,0)$ and for $t \to \infty$ end at the point $(T,R) = (\pi,0)$. The starting point is called *past timelike infinity* i^-, the ending point is called *future timilike infinity* i^+. ■

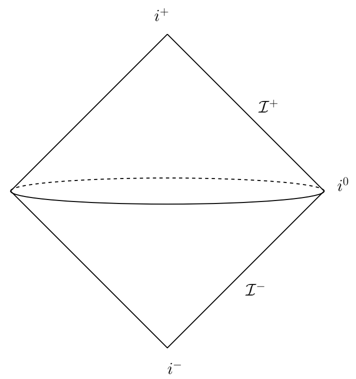


Figure 5.2: Unphysical Minkowski spacetime with coordinate θ suppressed.

Exercise 5.2 Show that (spacelike) lines of constant t in Minkowski spacetime mapped to $\hat{M}$ are curves which for $r = 0$ start at the point $(T, 0)$, where T is given by (5.16), and for $r \to \infty$ end at the point $(T, R) = (0, \pi)$. This point is called *spatial infinity* i^0. ■

Notice that with respect to metric $\hat{g}$ given by (5.13), i^0 is in fact a sphere of a zero radius, i.e. just a point.

Exercise 5.3 Now consider light rays. Outgoing null lines in Minkowski spacetime can be parametrized by $t = r + \text{const.}$, which means that such light rays are lines of constant $u(U)$. Similarly, ingoing rays are lines of constant v (V). Show that in $\hat{M}$, outgoing light rays extended to past/future infinity are straight lines intersecting surfaces $V = \mp\pi/2$, while ingoing rays extended to past/future infinity are straight lines intersecting surfaces $U = \mp\pi/2$. The line $V = \pi/2$ is called *future null infinity* $\mathscr{I}^+$, the line $U = -\pi/2$ is *past null infinity*. These lines meet at spatial infinity i^0. ■

The figure 5.1 is misleading in the sense that the actual spacetime is 4-dimensional, so that each point inside the triangle is in fact a 2-sphere of radius $\sin^2 R$. Since we cannot plot a truly 4-dimensional picture, we have to suppress one dimension by, e.g., setting $\theta = \pi/2$ and rotate the whole triangle around the axis T, obtaining so the Figure 5.2. However, this figure is misleading for another reason [Ste93]. We have already seen that the spatial infinity i^0 is in fact a single point, while in Figure 5.2 it looks like a circle. More appropriate picture of the spatial infinity is illustrated in Figure 5.3. Here, i^0 is a point from which the future null cone $\mathscr{I}^+$ and past null cone $\mathscr{I}^-$ emanate.

In order to simplify the visualization, we usually plot the unphysical Minkowski spacetime as a section of Figure 5.2 in the plane $\phi = 0$ which coincides with the plane $\phi = \pi$. The trajectories discussed in the exercises above are plotted in such section in figure 5.4.

5.2 Asymptotic flatness

In the previous section we started with the Minkowski spacetime in spherical coordinates in which we have a good intuition what the infinity is and how it is described in those coordinates. Minkowski space is flat everywhere so it must be also asymptotically flat according to any reasonable definition. We have found that the analysis we did in specific coordinates can be reformulated in a coordinate-free way as we now summarize in the following definition [Ste93].

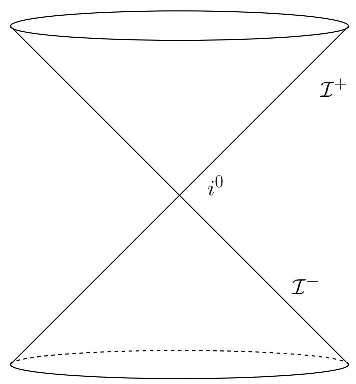
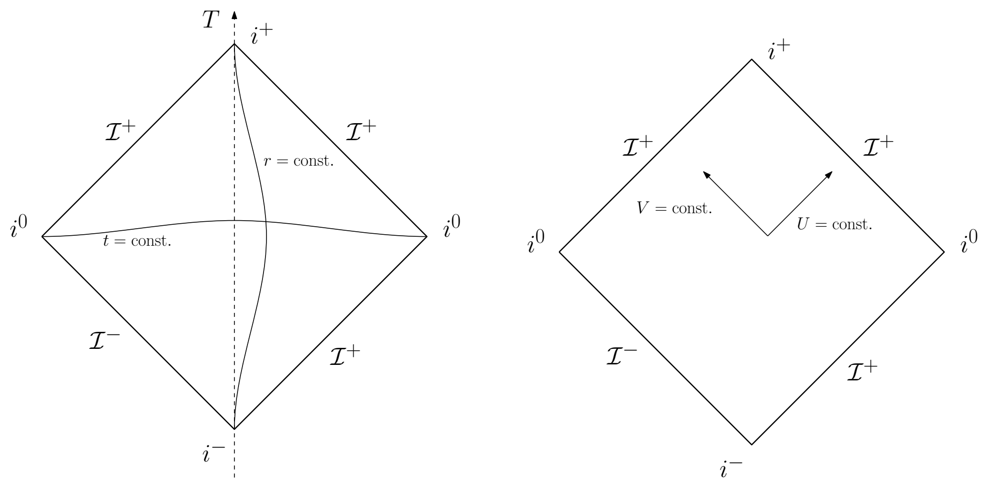


Figure 5.3: Appropriate visualization of causal structure near spatial infinity i^0.



a) Timelike and spacelike curves **b)** Null curves

Figure 5.4: Worldlines in Minkowski space mapped to the unphysical space.

Definition 5.2.1 Physical spacetime $(\tilde{M}, \tilde{g}_{ab})$ is called *asymptotically simple* if there exists another manifold (*unphysical spacetime*) (M, g_{ab}) and a mapping $\psi : \tilde{M} \mapsto M$ satisfying the following properties:

1. image $\Sigma = \psi(\tilde{M})$ is an open subset of M;
2. metrics of the two manifolds are related by a conformal rescaling
$$\psi^* g_{ab} = (\psi^* \Omega)^2 \, \tilde{g}_{ab}, \tag{5.18}$$
where $\Omega : M \mapsto \mathbb{R}$ is a real function on M;
3. $\Omega = 0$ on the boundary $\partial\Sigma$, but its gradient is non-vanishing there,
$$\Omega|_{\partial\Sigma} = 0, \qquad \mathrm{d}\Omega|_{\partial\Sigma} \neq 0. \tag{5.19}$$
4. the image of each null geodesic in $\tilde{M}$ has two endpoints on $\partial\Sigma$.

The unphysical spacetime (M, g_{ab}) is sometimes called the *asymptote* of the physical spacetime $(\tilde{M}, \tilde{g}_{ab})$.

R Here we denote the physical spacetime by $\tilde{M}$ and the unphysical one by M. The reason is that in what follows we will work mostly in the unphysical spacetime and we wish to omit the tildas as much as possible.

R For notational convenience, we will usually omit the mapping ψ (and pull-back ψ^*) from the equations. We will assume that $\psi : \tilde{M} \mapsto \Sigma$ is a bijection and therefore we can drag tensor fields form $\tilde{M}$ to M (and vice versa) freely. Hence, we simply write $g_{ab} = \Omega^2 \, \tilde{g}_{ab}$. For the same reason we can use the same abstract indices for tensor fields living on $\tilde{M}$ and M. Also we will not distinguish between $\Sigma = \psi(\tilde{M})$ and $\tilde{M}$ itself and we will freely use $\partial\tilde{M}$ for the boundary of Σ.

R In Equation (5.19) we use the symbol $\mathrm{d}\Omega$ for the gradient, although in the abstract index formalism we would prefer the notation $\nabla_a \Omega$. We did not specify the connection on M so far, hence we use the connection-independent version. Nevertheless, the object $\nabla_a \Omega$ does not depend on a specific choice of the connection, since Ω is a scalar function.

R Condition 3 can be easily verified for the conformal factor (5.10). More generally, the purpose of this requirement is to identify the infinty of the physical spacetime in a coordinate independent way: infinity of the spacetime is the set of points where the conformal factor vanishes. In the other hand, we will use the conformal factor to construct a coordinate system in the neighborhood of infinity. For that we need that Ω vanishes at infinity, but it is non-zero in the neighborhood, in other words, gradient of Ω must be non-zero so that Ω can be used as a coordinate.

While the requirement 4 of Definition 5.2.1 works for the Minkowski spacetime, it is too strong for spacetimes of physical interest. Indeed, in spacetimes containing black holes there will exist null geodesics which emanate from $\mathscr{I}^-$ but end up in singularity, and hence never reach $\mathscr{I}^+$. The purpose of this condition is to ensure that null infinity can be reached by *some* null geodesics. In order to include black hole spacetimes, we have to weaken the definition.

Definition 5.2.2 Spacetime $(\tilde{M}, \tilde{g})$ is called *weakly asymptotically simple* if there exists asymptoticaly simple spacetime $(\tilde{N}, \tilde{h}_{ab})$ with the asymptote (M, g_{ab}) such that there is an open neighborood $U \subset \tilde{N}$ of $\partial \tilde{N}$ which is isometric to an open set in $\tilde{M}$. Asymptote (M, g_{ab}) is then by definition also the *asymptote* of $\tilde{M}$.

R The last definition can be rephrased as follows: spacetime is weakly asympotically flat, if the neighborhood of its infinity coincides with a neighborhood of some spacetime which is asymptotically simple in a strong sense.

R Weakly asymptotically simple spacetimes will be referred to as *asymptotically flat spacetimes*. For mathematical purposes it is important to distinguish asymptotically simple and weakly asymptotically simple spacetimes but from a physical point of view, both definitions capture the physical intuition that spacetime far from the isolated source is flat.

R In [Ste93], the Definition 5.2.1 includes also condition on the Ricci tensor of physical spacetime near infinity. In this text, we will specify these conditions later.

R Notice that the choice of the conformal factor is not unique. Indeed, if (M, g_{ab}) is an asymptote of $(\tilde{M}, \tilde{g}_{ab})$ where $g_{ab} = \Omega^2 \tilde{g}_{ab}$, then any conformal factor of the form

$$\hat{\Omega} = \theta \, \Omega, \tag{5.20}$$

where θ is a strictly positive function will again satisfy the requirement 3 of Definition 5.2.1. We call the rescaling of the conformal factor of the form (5.20) the *gauge freedom* in the choice of the conformal factor.

5.3 Conformal transformations

In order to analyze asymptotic properties of an asymptotically flat physical spacetime $\tilde{M}$ it is convenient to work in its asymptote M, since there the infinity of $\tilde{M}$ is represented by a boundary of subset $\tilde{M}$, but otherwise this infinity is a well-defined set with regular metric where nothing special happens. Hence, we can analyze the properties of the infinity by local techniques. In this section we therefore introduce the connection ∇_a on M and associated curvature tensors. We will work mostly in the spinor formalism in which many formulas look simpler.

5.3.1 Covariant derivative

We consider the conformal rescalings of the metric, i.e. the transformations of the metric in the form

$$g_{ab} = \Omega^2 \, \tilde{g}_{ab}, \qquad g^{ab} = \Omega^{-2} \, \tilde{g}^{ab}. \tag{5.21}$$

Since the spinor equivalent of the metric is the symplectic form ε_{AB},

$$g_{ab} = \varepsilon_{AB} \, \varepsilon_{A'B'}, \tag{5.22}$$

and since Ω is a real function by definition, we have to prescribe the conformal transformation of the symplectic form by relations

$$\varepsilon_{AB} = \Omega \, \tilde{\varepsilon}_{AB}, \qquad \varepsilon^{AB} = \Omega^{-1} \, \varepsilon^{AB}. \tag{5.23}$$

First we need to define the connection in the unphysical spacetime. We assume that the connection $\tilde{\nabla}_a$ in the physical spacetime is the usual Levi-Civita connection satisfying $\nabla_a g_{bc} = 0$. In the spinor form,

$$\tilde{\nabla}_{AA'}\tilde{\varepsilon}_{BC} = 0. \tag{5.24}$$

Analogously, in the unphysical spacetime we define the Levi-Civita connection ∇_a compatible with g_{ab},

$$\nabla_{AA'}\varepsilon_{BC} = 0. \tag{5.25}$$

Similarly to the tensor analysis, any two connections are related by

$$\tilde{\nabla}_{AA'}\xi_B = \nabla_{AA'}\xi_B + \Upsilon_{BA'}\,\xi_A. \tag{5.26}$$

The form of $\Upsilon_{AA'}$ is found in a way completely analogous to the tensorial case. Imposing (5.23) and (5.25), we have

$$\nabla_{AA'}\varepsilon_{BC} = \tilde{\varepsilon}_{BC}\,\nabla_{AA'}\Omega + \Omega\left(\tilde{\nabla}_{AA'}\tilde{\varepsilon}_{BC} - \Upsilon_{BA'}\,\tilde{\varepsilon}_{AC} - \Upsilon_{CA'}\,\tilde{\varepsilon}_{BA}\right). \tag{5.27}$$

Using (5.24) we find

$$\tilde{\varepsilon}_{BC}\nabla_{AA'}\Omega = \Omega\,(\Upsilon_{BA'}\,\tilde{\varepsilon}_{AC} + \Upsilon_{CA'}\,\tilde{\varepsilon}_{BA}). \tag{5.28}$$

Next we rewrite the last equation with cyclic permutations of indices (ABC, BCA, CAB), obtaining three equations. We add the first two of them and subtract the third:

$$2\,\Omega\,\Upsilon_{BA'}\,\tilde{\varepsilon}_{AC} = \tilde{\varepsilon}_{BC}\,\nabla_{AA'}\Omega - \tilde{\varepsilon}_{CA}\,\nabla_{BA'}\Omega + \tilde{\varepsilon}_{AB}\,\nabla_{CA'}\Omega. \tag{5.29}$$

Contracting with $\tilde{\varepsilon}^{AC}$ we finally find

$$\Upsilon_{AA'} = \nabla_{AA'}\ln\Omega. \tag{5.30}$$

Since the mapping ψ identifies the physical spacetime and the (portion of the) unpysical one, for a scalar function it does not matter if we regard it as a function on $\tilde{M}$ or a function on M. Hence, we naturally require that for scalars the covariant derivatives $\tilde{\nabla}_a$ and ∇_a coincide, i.e. $\tilde{\nabla}_a f = \nabla_a f$ for any scalar. If we additionally impose that the unphysical derivative ∇_a satisfy the Leibniz rule, we can also define the action of ∇_a on a contravariant spinor μ^A. Let ξ_A and μ^A be arbitrary spinors, then

$$\nabla_{AA'}(\xi_B\,\mu^B) = \tilde{\nabla}_{AA'}(\xi_B\,\mu^B). \tag{5.31}$$

On the one hand we have

$$\nabla_{AA'}(\xi_B\,\mu^B) = \xi_B\nabla_{AA'}\mu^B + \mu^B\nabla_{AA'}\xi_B, \tag{5.32}$$

on the other hand we have

$$\tilde{\nabla}_{AA'}(\xi_B\,\mu^B) = \xi_B\tilde{\nabla}_{AA'}\mu^B + \mu^B\,(\nabla_{AA'}\xi_B + \Upsilon_{BA'}\,\xi_A), \tag{5.33}$$

so by comparing the last two equations we find

$$\xi_B\nabla_{AA'}\mu^B = \xi_B\tilde{\nabla}_{AA'}\mu^B + \Upsilon_{CA'}\,\xi_A\,\mu^C. \tag{5.34}$$

Using the identity $\xi_A = \varepsilon_A{}^B\,\xi_B$ and the fact that ξ_B is arbitrary, we finally arrive at the relation

$$\tilde{\nabla}_{AA'}\mu^B = \nabla_{AA'}\mu^B - \Upsilon_{CA'}\,\varepsilon_A{}^B\,\mu^C. \tag{5.35}$$

These simple calculations lead to the following definition.

Definition 5.3.1 Connection ∇_a on the unphysical spacetime (M, g_{ab}) which is an asymptote of asymptotically flat spacetime $(\tilde{M}, \tilde{g}_{ab}$ with Levi-Civita connection $\tilde{\nabla}_a$ is defined by relations

1. $\nabla_a f = \tilde{\nabla}_a f$ for any scalar function f;
2. $\nabla_{AA'}\xi_B = \tilde{\nabla}_{AA'}\xi_B - \Upsilon_{BA'}\,\xi_A$, where $\Upsilon_a = \nabla_a \ln\Omega$, for any spinor ξ_B;
3. $\nabla_{AA'}\mu^B = \tilde{\nabla}_{AA'}\mu^B + \Upsilon_{CA'}\,\varepsilon_A{}^B\,\mu^C$;
4. $\nabla_{AA'}$ is real, i.e. $\overline{\nabla_{AA'}} = \nabla_{AA'}$;
5. ∇_a satisfies the Leibniz rule when acting on a direct product.

Exercise 5.4 Show that the action of ∇_a on a co-vector α_a reads

$$\nabla_a\alpha_b = \tilde{\nabla}_a\alpha_b - \alpha_a\,\Upsilon_b - \alpha_b\,\Upsilon_a + g_{ab}\,\alpha_c\,\Upsilon^c, \tag{5.36}$$

where, of course, $\Upsilon_a = \Upsilon_{AA'}$. (Hint: consider a co-vector in the form $\alpha_a = \kappa_A\,\bar{\mu}_{B'}$, generalization to arbitrary co-vector is trivial.) ■

5.3.2 Zero rest mass equations

In spinor formalism, general massless field of spin s is described by a spinor $\phi_{A\ldots B}$ with $2s$ indices. In the source-free case, such fields satisfy the so-called *zero rest mass equation* (ZRM) in the physical spacetime

$$\tilde{\nabla}^A_{A'}\tilde{\phi}_{A\ldots B} = 0, \tag{5.37}$$

where $\phi_{A\ldots B}$ is a totally symmetric spinor. In general, massless fields are conformally invariant, because there is no natural length or time scale: with a massless particle we cannot associate a frame of reference. Let us discuss the conformal invariance in the context of present formalism.

Definition 5.3.2 A quantity X (tensorial or spinorial) is said to have a *conformal weight* w if under the conformal rescaling of the metric, $g_{ab} = \Omega^2\tilde{g}_{ab}$, it transforms as

$$\tilde{X} = \Omega^w X. \tag{5.38}$$

We will now show that the ZRM equation (5.37) is conformally invariant, if $\tilde{\phi}_{A\ldots B}$ has conformal weight 1. Suppose that

$$\tilde{\phi}_{AB\ldots C} = \Omega\,\phi_{AB\ldots C}. \tag{5.39}$$

Then,

$$\begin{aligned}\tilde{\nabla}^A_{A'}\tilde{\phi}_{AB\ldots C} &= \tilde{\varepsilon}^{AX}\tilde{\nabla}_{XA'}\tilde{\phi}_{AB\ldots C} = \Omega^{-1}\varepsilon^{AX}\tilde{\nabla}_{XA'}\left(\Omega\,\phi_{AB\ldots C}\right) = \\ &= \Omega^{-1}\,\phi_{AB\ldots C}\nabla^A_{A'}\Omega + \varepsilon^{AX}\left(\nabla_{XA'}\phi_{AB\ldots C} + \Upsilon_{AA'}\,\phi_{XB\ldots C} + \Upsilon_{BA'}\,\phi_{AX\ldots C} + \cdots + \Upsilon_{CA'}\,\phi_{AB\ldots X}\right).\end{aligned} \tag{5.40}$$

Since the spinor $\phi_{AB\ldots C}$ is symmetric, contraction of ε^{AX} with $\phi_{\ldots A\ldots X\ldots}$ yields zero, which eliminates all terms in the bracket except for the first two. Thus,

$$\begin{aligned}\tilde{\nabla}^A_{A'}\tilde{\phi}_{AB\ldots C} &= \Omega^{-1}\,\phi_{AB\ldots C}\nabla^A_{A'}\Omega + \nabla^A_{A'}\phi_{AB\ldots C} - \varepsilon^{XA}\,\Upsilon_{AA'}\,\phi_{XB\ldots C} = \\ &= \Omega^{-1}\,\phi_{AB\ldots C}\nabla^A_{A'}\Omega + \nabla^A_{A'}\phi_{AB\ldots C} - \Omega^{-1}\,\phi_{XB\ldots C}\nabla^X_{A'}\Omega = \nabla^A_{A'}\phi_{AB\ldots C}.\end{aligned} \tag{5.41}$$

Hence, the unphysical ZRM field satisfies the equation

$$\nabla^A_{A'}\phi_{AB\ldots C} = 0 \tag{5.42}$$

and therefore equation (5.37) is conformally invariant, provided that the conformal weight of $\tilde{\phi}_{AB\ldots C}$ is $w = 1$.

5.3.3 Transformations of curvature spinors

Recall that in the spinor formalism we decompose all spinors into irreducible parts: totally symmetric part and antisymmetric part which is always proportional to ε_{AB}. This applies also to spinor equivalents of tensors. If the tensors possess some additional symmetry, the general spinor decomposition simplifies. In the case of the Riemann tensor, we have the following symmetries (see [PR84; Wal84]):

1. $R_{abcd} = R_{[ab][cd]}$;
2. $R_{abcd} = R_{cdab}$;
3. $R_{[abc]d} = 0$.

These symmetries imply that the spinor decomposition of the Riemann tensor is

$$\begin{aligned} R_{abcd} = \Psi_{ABCD}\,\varepsilon_{A'B'}\,\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\,\varepsilon_{AB}\,\varepsilon_{CD} + \\ + \Phi_{ABC'D'}\,\varepsilon_{A'B'}\,\varepsilon_{CD} + \Phi_{CDA'B'}\,\varepsilon_{C'D'}\,\varepsilon_{AB} - \\ - 2\Lambda\left(\varepsilon_{AB}\,\varepsilon_{CD}\,\varepsilon_{A'(C'}\,\varepsilon_{D')B'} + \varepsilon_{A'B'}\,\varepsilon_{C'D'}\,\varepsilon_{A(C}\,\varepsilon_{D)B}\right). \end{aligned} \tag{5.43}$$

The first part is the Weyl tensor,

$$C_{abcd} = \Psi_{ABCD}\,\varepsilon_{A'B'}\,\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\,\varepsilon_{AB}\,\varepsilon_{CD}, \tag{5.44}$$

where Ψ_{ABCD} is completely symmetric Weyl spinor. The second part is given by the real spinor $\Phi_{ABC'D'}$ which is related to the trace-free part of the Ricci tensor,

$$-2\,\Phi_{ABA'B'} = R_{ab} - \frac{1}{4}\,g_{ab}\,R, \tag{5.45}$$

and the trace of the Riemann tensor is carried by the scalar Λ related to the scalar curvature by

$$\Lambda = \frac{1}{24}R. \tag{5.46}$$

In the spinor formalism, Einstein's equations acquire the form

$$\Phi_{ABA'B'} = 4\pi\,T_{(AB)(A'B')}, \qquad 3\Lambda = T_a^{\ a}. \tag{5.47}$$

Hence, Einstein's equations are just algebraic relations between the curvature spinors and the energy-momentum tensor. The role of the field equations is now played by the identities satisfied by the Riemann tensor. The first one is the so-called *Ricci identity*, which is usually regarded as a definition of the Riemann tensor,

$$(\nabla_c\nabla_d - \nabla_d\nabla_c)X^a = -R^a{}_{bcd}X^b. \tag{5.48}$$

The spinor form of the Ricci identity is

$$\Box_{AB}\xi_C = \Psi_{ABCD}\,\xi^D - 2\Lambda\,\xi_{(A}\varepsilon_{B)C}, \qquad \Box_{A'B'}\xi_C = \xi^D\,\Phi_{CDA'B'}. \tag{5.49}$$

The second equation is the Bianchi identity

$$\nabla_{[p}R_{ab]cd} = 0 \tag{5.50}$$

which translates to

$$\nabla^A_{A'}\Psi_{ABCD} = \nabla^{B'}_{(B}\Phi_{CD)A'B'}, \qquad \nabla^{AA'}\Phi_{ABA'B'} = -3\nabla_{BB'}\Lambda. \tag{5.51}$$

Under conformal transformation, the irreducible parts of the Riemann tensor transform according to formulas

$$\tilde{\Psi}_{ABCD} = \Psi_{ABCD}, \tag{5.52a}$$

$$\tilde{\Phi}_{ABA'B'} = \Phi_{ABA'B'} + \Omega^{-1}\,\nabla_{A(A'}\nabla_{B')B}\Omega, \tag{5.52b}$$

$$\tilde{\Lambda} = \Omega^2\Lambda - \frac{1}{4}\,\Omega\Box\Omega + \frac{1}{2}\,(\nabla_a\Omega)(\nabla^a\Omega). \tag{5.52c}$$

Exercise 5.5 Use the Ricci and Bianchi identities to derive formulas (5.52). ■

5.4 Properties of null infinity

Properties of the null infinity depend $\mathscr{I}$ crucially on the presence of matter in its neighborhood. Since we are considering asymptotically flat spacetimes, we expect that there is no source close to infinity, but even isolated sources can emit radiation or produce fields that extend to infinity. It is then important how fast these fields decay as one approaches $\mathscr{I}$. The conformal factor Ω can be conveniently used as a coordinate near $\mathscr{I}$. Moreover, we will usually assume that all quantities in the unphysical spacetime are regular on $\mathscr{I}$.

Definition 5.4.1 We will use the notation

$$X = \mathscr{O}(\Omega^n) \tag{5.53}$$

whenever the quantity X can be expanded near $\mathscr{I}$ into the Taylor series of the form

$$X = X^0\,\Omega^n + X^1\,\Omega^{n+1} + \cdots . \tag{5.54}$$

In particular, quantity $X = \mathscr{O}(1)$ if it is non-vanishing and regular on $\mathscr{I}$. We will employ the notation $X \doteq Y$ if the two quantities are equal on $\mathscr{I}$ but not necessarily elswhere. For example, $\Omega \doteq 0$.

In the case of Minkowski space we have seen that $\mathscr{I}$ is in fact a null hypersurface. This can change in the presence of matter or cosmological constant (which we do not consider here), but with sufficiently fast fall-off of the field $\mathscr{I}$ is null hypersurface even the presence of matter near infinity. We will start with the simplest observation.

R We have already mentioned that we assume the regularity of unphysical quantities on $\mathscr{I}$. In particular, for now we assume that $g_{ab} = \mathscr{O}(1)$, i.e. the unphysical metric is non-vanishing and non-singular on $\mathscr{I}$. Next we assume that both g_{ab} and Ω have the smoothness C^2 across $\mathscr{I}$, i.e. both objects are continuously twice differentiable. This implies that unphysical curvature tensor R_{abcd} (and its spinorial parts $\Psi_{ABCD}, \Phi_{ABA'B'}, \Lambda$) are $\mathscr{O}(1)$ on $\mathscr{I}$ as well.

Theorem 5.4.1 Suppose that physical scalar curvature $\tilde{\Lambda}$ vanishes at $\mathscr{I}$ (or, equivalently, the trace $\tilde{T}^a_a$ of physical energy-momentum tensor vanishes at $\mathscr{I}$). Then $\mathscr{I}$ is a null hypersurface.

Proof. Let us define $n_a = -\nabla_a \Omega$, so that n_a is orthogonal to hypersurfaces $\Omega = \text{const.}$, in particular it is orthogonal to $\mathscr{I}$. Relation (5.52c) then acquires the form

$$\tilde{\Lambda} = \Omega^2 \Lambda + \frac{1}{4}\Omega \nabla_a n^a + \frac{1}{2} n_a n^a. \tag{5.55}$$

Physical curvature $\tilde{\Lambda}$ vanishes at $\mathscr{I}$ by assumption. Terms proportional to Ω vanish on $\mathscr{I}$ because of assumed regularity of $\nabla_a \Omega$ and Λ. Hence, the last equation restricted to $\mathscr{I}$ immediately yields

$$n_a n^a \doteq 0. \tag{5.56}$$

■

R In other words, $n^a n_a = \mathscr{O}(\Omega)$ and therefore can be written in the form

$$n^a n_a \equiv (\nabla_a \Omega)(\nabla^a \Omega) = \Omega f \qquad \text{for some} \quad f = \mathscr{O}(1). \tag{5.57}$$

Theorem 5.4.2 On $\mathscr{I}$, vector n^a is a null geodesics.

Proof. We have

$$n^a \nabla_a n_b = (\nabla^a \Omega)(\nabla_a \nabla_b \Omega) = (\nabla^a \Omega)(\nabla_b \nabla_a \Omega) = \frac{1}{2}\nabla_b ((\nabla_a \Omega)(\nabla^a \Omega)) = \\ = \frac{1}{2}\nabla_b (n_a n^a) = \frac{1}{2}\nabla_b (\Omega f) = -\frac{1}{2} f n_b + \frac{1}{2}\Omega \nabla_b f \doteq -\frac{1}{2} f n_b. \quad (5.58)$$

Here we have used the vanishing of the torsion ($[\nabla_a, \nabla_b]\Omega = 0$) and relation (5.57) and we restricted the whole expression to $\mathscr{I}$ in the last step (symbol $\doteq$). Thus, $n^a \nabla_a n_b$ is proportional to n_b and therefore n_b is a geodesics. ∎

R Notice that, according to the transport equation (3.8e), n^a is geodesic if the spin coefficient ν vanishes,

$$\nu \doteq 0; \quad (5.59)$$

here we used symbol $\doteq$, since n^a is not necessarily geodesic away from $\mathscr{I}$.

R Theorem 5.4.2 shows that $\mathscr{I}$ has topology $\mathbb{R} \times K$ where K are "slices" of $\mathscr{I}$ transversal to the orbits of the null generator n^a. The following theorem shows that K is in fact a topological sphere [PR86].

Theorem 5.4.3 In asymptotically simple spacetimes for which $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ is everywhere a null hypersurface, the topology of the past and future null infinity is

$$\mathscr{I}^+ \cong \mathscr{I}^- \cong \mathbb{R} \times \mathscr{S}^2, \quad (5.60)$$

where $\mathscr{S}^2$ is two dimensional topological sphere.

Proof. For proof, see [Pen65]. ∎

IV Quasi-local quantities

$$\nabla_A^{A'}\psi_{A'B'C'D'} = 0$$

$$\phi_{AB} = \phi_{(AB)} + \frac{1}{2}\epsilon_{AB}\,\phi_X{}^X$$

$$C_{abcd} = \psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \text{c.c.}$$

6. Geometry of space-like 2-surfaces

This chapter is mainly based on [Sza]. Consider a space-time M with metric g_{ab} and a two-dimensional space-like submanifold $\mathscr{S}$. That is, at each point $P \in \mathscr{S}$ there exists a unit time-like normal t^a and a space-like unit normal v^a, thereby satisfying

$$t^a t_a = -v^a v_a = 1, \qquad t^a v_a = 0, \qquad t^a, v^a \text{ orthogonal to } \mathscr{S}. \tag{6.1}$$

Using these two normals we can form two *null* vectors

$$\ell^a = \frac{1}{\sqrt{2}}(t^a + v^a), \qquad n^a = \frac{1}{\sqrt{2}}(t^a - v^a), \tag{6.2}$$

which satisfy the usual Newman–Penrose normalization $\ell^a n_a = 1$. We will also assume

$$\ell^a = o^A \bar{o}^{A'}, \qquad n^a = \iota^A \bar{\iota}^{A'}. \tag{6.3}$$

Choosing such (non-unique) spinors o^A and ι^A, the basis which is tangent to $\mathscr{S}$ is $m^a = o^A \bar{\iota}^{A'}$ and its complex conjugate.

Definition 6.0.1 *Projector* onto the tangent space of $\mathscr{S}$ with normals t^a and v^a satisfying (6.1) is a tensor defined by

$$\Pi^a_b = \delta^a_b - t^a t_b + v^a v_b = \delta^a_b - \ell^a n_b - n^a \ell_b. \tag{6.4}$$

Tensor $\tau^{a\ldots b}_{c\ldots d}$ is called *surface tensor* if

$$\Pi^a_e \cdots \Pi^b_f \Pi^r_c \ldots \Pi^s_d \, \tau^{e\ldots f}_{r\ldots s} = \tau^{a\ldots b}_{c\ldots b}. \tag{6.5}$$

The choice of normals t^a and v^a satisfying (6.1) is of course not unique. Form of the projector (6.4) suggests the most general transformation (called *gauge transformation*) which preserves Π^a_b.

Theorem 6.0.1 Let $\mathscr{S}$ be a space-like 2-submanifold of space-time M, let t^a and v^a satisfy conditions (6.1), and let ℓ^a and n^a be given by (6.2). Then, the projector (6.4) is invariant under the gauge transformation parametrized by single real parameter A

$$\ell^a \mapsto A\,\ell^a, \qquad n^a \mapsto A^{-1}\,n^a, \tag{6.6}$$

or, in terms of t^a and v^a,

$$t^a \mapsto t^a \cosh\varphi + v^a \sinh\varphi, \qquad v^a \mapsto t^a \sinh\phi + v^a \cosh\phi, \tag{6.7}$$

where

$$\tanh\phi = \frac{A^2-1}{A^2+1}. \tag{6.8}$$

Proof. Proving Eq. (6.6) is trivial, as this transformation leaves $\ell \otimes n$ invariant and hence also the projector. Re-expressing t^a and v^a as

$$t^a = \frac{1}{\sqrt{2}}\left(\ell^a + n^a\right), \qquad v^a = \frac{1}{\sqrt{2}}\left(\ell^a - n^a\right), \tag{6.9}$$

and performing the transformation (6.6) we arrive at

$$t^a \mapsto \underbrace{\frac{A+A^{-1}}{2}}_{\alpha}\,t^a + \underbrace{\frac{A-A^{-1}}{2}}_{\beta}\,v^a. \tag{6.10}$$

Obviously, $\alpha^2 - \beta^2 = 1$ which means we can write $\alpha = \cosh\phi$, $\beta = \sinh\phi$, from which (6.8) follows. Similarly for v^a. ■

By TM we denote the tangent bundle of the space-time. Its *restriction*[1] to $\mathscr{S}$ will be denoted by $V(\mathscr{S})$. Hence, locally, points of $V(\mathscr{S})$ are of the form (P, u^a), where $P \in \mathscr{S}$ and $u^a \in V^a_P(\mathscr{S})$. Here, $V^a_P(\mathscr{S})$ is a fiber of bundle $V^a(\mathscr{S})$ at point P and it is isomorphic to $T_P M$. At any point $P \in \mathscr{S}$ we can split the fiber $V_P(\mathscr{S})$ into a direct sum

$$V_P(\mathscr{S}) = N_P(\mathscr{S}) \oplus T_P(\mathscr{S}), \tag{6.11}$$

where $N_P(\mathscr{S})$ is a subspace of vectors normal to $\mathscr{S}$ and $T_P(\mathscr{S})$ is a subspace of vectors tangent to $\mathscr{S}$. That is, any $u^a \in V^a_P(\mathscr{S})$ is decomposed as

$$u^a = \alpha\, n^a + \beta\, \tau^a, \tag{6.12}$$

where $n^a \in N^a_P(\mathscr{S})$ and $\tau^a \in T^a_P(\mathscr{S})$. Hence, the total bundle $V(\mathscr{S})$ can be itself written as the direct sum of sub-bundles

$$V(\mathscr{S}) = N(\mathscr{S}) \oplus T(\mathscr{S}), \tag{6.13}$$

where $N(\mathscr{S})$ is a *normal sub-bundle* and $T(\mathscr{S})$ is the *tangent sub-bundle*. These sub-bundles are mutually orthogonal with respect to the fiber metric g_{ab} [2].

[1] General bundle is denoted by $\pi : E \mapsto B$, where π is the projection, E is the total space and B is the base manifold. For $B' \subset B$ we define the *restriction* of bundle $\pi : E \mapsto B$ to B' as a bundle $\pi : E' \mapsto B'$, where $E' = \pi^{-1}(B')$.

[2] We say that the space-time metric g_{ab} is *fiber metric*, because fibers of $V(\mathscr{S})$ are isomorphic to tangent spaces of M, on which metric g_{ab} is defined.

Definition 6.0.2 The *Lorentzian vector bundle* is a triple $(V^a(\mathscr{S}), g_{ab}, \Pi^a_b)$, where $V^a \mapsto \mathscr{S}$ is a vector bundle over a surface $\mathscr{S}$, equipped with the fiber metric g_{ab} and the projection $\Pi^a_b : V^b(\mathscr{S}) \mapsto T^a(\mathscr{S})$ onto the tangent sub-bundle of $V^a(\mathscr{S})$.

R The projector Π^a_b can be also regarded as the metric on the tangent sub-bundle. This induced metric is defined by

$$q_{ab} = g_{ac}\Pi^c_b = g_{ab} - 2\,\ell_{(a}n_{b)} = -2\,m_{(a}\bar{m}_{b)}. \tag{6.14}$$

Similarly, on the normal sub-bundle we have the induced metric

$$^{\perp}q_{ab} = g_{ab} - q_{ab} = 2\,\ell_{(a}n_{b)}. \tag{6.15}$$

Next, in the fibers of $V(\mathscr{S})$ with metric g_{ab} we have automatically also the volume form

$$\varepsilon_{abcd} = 24\,\ell_{[a}n_b m_c \bar{m}_{d]}. \tag{6.16}$$

This expression is fixed by total antisymmetry, by choosing the orientation $\varepsilon_{abcd}\ell^a n^b m^c \bar{m}^d = 1$ and by $\varepsilon^{abcd}\varepsilon_{abcd} = 4! = 24$. Induced volume form on $T(\mathscr{S})$ is then

$$\varepsilon_{ab} = \varepsilon_{abcd}\ell^c n^d = -2\,m_{[c}\bar{m}_{d]}, \qquad \varepsilon_{ab}m^a\bar{m}^b = 1, \qquad \varepsilon_{ab}\varepsilon^{ab} = -2. \tag{6.17}$$

Similarly, induced volume form on $N(\mathscr{S})$ is

$$^{\perp}\varepsilon_{ab} = m^c\bar{m}^d\varepsilon_{cdab} = -2\,\ell_{[a}n_{b]}, \qquad {}^{\perp}\varepsilon_{ab}\ell^a n^b = 1, \qquad {}^{\perp}\varepsilon_{ab}{}^{\perp}\varepsilon^{ab} = -2. \tag{6.18}$$

All these objects are gauge invariant.

R We can also establish the relation between surface tensors and spinors. Let $S^A(\mathscr{S})$ be a bundle of spinors on M restricted to $\mathscr{S}$ and let $S^{A'}(\mathscr{S})$ be the bundle of primed spinors. Then, we can construct a bundle $S^A(\mathscr{S})\otimes S^{A'}(\mathscr{S})$ which can be identified with complexified Lorentzian vector bundle. In particular, we have the identification of Hermitian spinors with real vectors through the soldering form $\sigma^a_{AA'}$,

$$\text{Hermitian sub-bundle of } S^A\otimes S^{A'} \mapsto V^a : \kappa^{AA'} \mapsto \sigma^a_{AA'}\kappa^{AA'}. \tag{6.19}$$

6.1 Connections on the Lorentzian bundle

On the Lorentzian vector bundle $(V(\mathscr{S}), g_{ab}, \Pi^a_b)$ we can define two meaningful connections. Since each fiber of $V(\mathscr{S})$ is equipped with the metric g_{ab}, there is a natural action of group $O(1,3)$ on $V(\mathscr{S})$ preserving the fiber metric. In addition to that, the projection Π^a_b selects preferred sub-bundles $T(\mathscr{S})$ and $N(\mathscr{S})$. On the former, the transformation preserving the induced metric q_{ab} is just the *spin*

$$m^a \mapsto e^{\mathrm{i}\theta}m^a, \qquad \bar{m}^a \mapsto e^{-\mathrm{i}\theta}\bar{m}^a, \tag{6.20}$$

i.e. the rotation in the space-like plane spanned by m^a and $\bar{m}^a$; hence, natural group on $T(\mathscr{S})$ is $O(2)$. On the normal bundle, the gauge transformation preserving $^{\perp}q_{ab}$ is the boost (6.6), so the group acting on $N(\mathscr{S})$ is $O(1,1)$.

Definition 6.1.1 *Intrinsic covariant derivative* δ_a on $V(\mathscr{S})$ is defined by

$$\delta_a X_b = \Pi_a^c \Pi_b^d \nabla_c X_d. \tag{6.21}$$

R This definition extends naturally to action of δ_a on tensor of arbitrary rank:

$$\delta_a \tau^{b\ldots c}_{d\ldots e} = \Pi_a^f \Pi_r^b \ldots \Pi_s^c \Pi_d^p \ldots \Pi_e^q \nabla_f \tau^{r\ldots s}_{p\ldots q}. \tag{6.22}$$

That is, we first calculate usual covariant derivative and then project all indices via the projector Π^a_b.

R Clearly, δ_a annihilates both the fiber metric g_{ab}. However, it also annihilates the induced 2-metric q_{ab}, for we have

$$\delta_a q_{bc} = \Pi_a^p \Pi_b^r \Pi_c^s \nabla_p (g_{rs} - \ell_r n_s - \ell_s n_r) = 0, \tag{6.23}$$

since $\nabla_p g_{rs} = 0$, $\Pi_b^r \ell_r = 0$, $\Pi_c^s n_s = 0$. Hence, δ_a is the Levi-Civita connection for the metric q_{ab}.

In usual 3+1 formulation of general relativity, we foliate the space-time by hypersurfaces with unit normal n^a and define the projector $h_a^b = \delta_a^b - n_a n^b$ and corresponding intrinsic derivative $\mathscr{D}_a$. From this we can calculate the intrinsic curvature of the hypersurface. On the other hand, the extrinsic curvature which describes how the hypersurface is embedded in the space-time is defined as $K_{ab} = \mathscr{D}_a n_b$.

In the present setting, we have two normals, t^a and v^a (or, equivalently, ℓ^a and n^a), and hence we have two extrinsic curvatures,

$$\tau_{ab} = \delta_a t_b, \qquad v_{ab} = \delta_a v_b. \tag{6.24}$$

Definition 6.1.2 The *Sen connection* Δ_a on $V(\mathscr{S})$ is defined by

$$\Delta_a X_b = \Pi_a^r \nabla_r X_b. \tag{6.25}$$

R This definition extends to tensors of arbitrary rank via

$$\Delta_a \tau^{b\ldots c}_{e\ldots f} = \Pi_a^d \nabla_d \tau^{b\ldots c}_{e\ldots f}. \tag{6.26}$$

That is, we calculate usual covariant derivative and then project just the index on the derivative.

In other words, both intrinsic derivative δ_a and the Sen derivative Δ_a can act on any vector $X^a \in V^a(\mathscr{S})$ which is not necessarily tangent. Both these connections evaluate the derivative in the tangential derivative, because in both cases we project the direction of the derivative by Π_a^b. However, the result of intrinsic differentiation is always a surface tensor (in the sense of definition 6.0.1), while the result of $\Delta_a X_b$ can have also a normal component. For example, we have

$$\Delta_a t_b = (\Pi_b^c + t_b t^c - v_b v^c) \Delta_a t_c = \delta_a t_b + t_b \underbrace{t^c \Delta_a t_c}_{=0} - v_b v^c \Delta_a t_c. \tag{6.27}$$

Hence, the normal component of the connection Δ_a can be encoded in the connection 1-form

$$A_a = v^c \Delta_a t_c. \tag{6.28}$$

Using this 1-form, we can write

$$\Delta_a t^b = \delta_a t^b - A_a v^b, \qquad \Delta_a v^b = \delta_a v^b - t^b A_a. \tag{6.29}$$

The extrinsic curvatures and the 1-form A_a are not gauge independent, as the following theorem shows.

Theorem 6.1.1 Let τ_{ab}, ν_{ab} and A_a be defined by

$$\tau_{ab} = \delta_a t_b, \qquad \nu_{ab} = \delta_a v_b, \qquad A_a = v^c \Delta_a t_c. \tag{6.30}$$

Then, under the boost (6.7), we have

$$\tau_{ab} \mapsto \cosh\phi\, \tau_{ab} + \sinh\phi\, \nu_{ab}, \quad \nu_{ab} \mapsto \sinh\phi\, \tau_{ab} + \cosh\phi\, \nu_{ab}, \quad A_a \mapsto A_a - \delta_a \phi. \tag{6.31}$$

Proof. The proof is simple but one has to be careful. Connection δ_a is defined through projector Π^a_b and, hence, gauge invariant. For τ_{ab} we have

$$\tau_{ab} = \delta_a t_b \mapsto \delta_a \left(\cosh\phi\, t_b + \sinh\phi\, v_b\right) = \Pi^c_a \Pi^d_b \left(t_d \sinh\phi\, \nabla_c \phi + \cosh\phi \nabla_c t_d\right) + \\ + \Pi^c_a \Pi^d_b \left(v_d \cosh\phi\, \nabla_c \phi + \sinh\phi\, \nabla_c v_d\right) = \cosh\phi\, \tau_{ab} + \sinh\phi\, \nu_{ab}, \tag{6.32}$$

and similarly for ν_{ab} and A_a. ■

Unlike δ_a, the Sen connection Δ_a *does not* annihilate the induced metric. Instead, we have

$$\Delta_a q_{bc} = 2\, \nu_{a(b} v_{c)} - 2\, \tau_{a(b} t_{c)}. \tag{6.33}$$

Connection δ_a is torsion-free, as can be easily checked. This is not true for Δ_a. First, we have

$$\Delta_a \Delta_b \phi = \Delta_a \left(\Pi^c_b \nabla_c \phi\right) = \left(\Delta_a \Pi^c_b\right) \nabla_c \phi + \Pi^c_b \Delta_a \nabla_c \phi. \tag{6.34}$$

Antisymmetrization in $[ab]$ yields

$$\Delta_{[a} \Delta_{b]} \phi = \Delta_{[a} \Pi^c_{b]} \nabla_c, \tag{6.35}$$

since $\Pi^c_b \Delta_a \nabla_c \phi = \Pi^c_b \Pi^f_e \nabla_f \nabla_c \phi$ is already symmetric in (ba), provided that ∇_a is torsion-free. Defining

$$Q^c_{ab} = \tau^e_a t_b - \nu^c_a v_b, \tag{6.36}$$

we can write the anti-commutator as

$$\Delta_{[a} \Delta_{b]} \phi = -Q^c_{[ab]} \nabla_c \phi, \tag{6.37}$$

showing that the torsion tensor of the connection Δ_a.

V

Appendices

$$\nabla_A^{A'}\psi_{A'B'C'D'} = 0$$

$$\phi_{AB} = \phi_{(AB)} + \frac{1}{2}\epsilon_{AB}\,\phi_X{}^X$$

$$C_{abcd} = \psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \text{c.c.}$$

A. Notation, conventions, abbreviations

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{R}^n$	The set of n-tuples of real numbers of the form $(x_1, \dots x_n) \in \mathbb{R}^n$
$\mathbb{C}^n$	The set of n-tuples of complex numbers of the form $(z_1, \dots z_n) \in \mathbb{C}^n$
$\mathbb{E}^n$	Euclidean space of dimension n

$\Lambda^p(L)$	Set of p-forms on vector space L

$$\nabla_A^{A'} \psi_{A'B'C'D'} = 0$$

$$\phi_{AB} = \phi_{(AB)} + \frac{1}{2} \epsilon_{AB} \phi_X{}^X$$

$$C_{abcd} = \psi_{ABCD} \epsilon_{A'B'} \epsilon_{C'D'} + \text{c.c.}$$

B. Spin weighted spherical harmonics

$$\nabla_A^{A'} \psi_{A'B'C'D'} = 0$$

$$\phi_{AB} = \phi_{(AB)} + \frac{1}{2}\epsilon_{AB}\phi_X{}^X$$

$$C_{abcd} = \psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \text{c.c.}$$

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